System Reliability Estimation under Prior-Data Conflict

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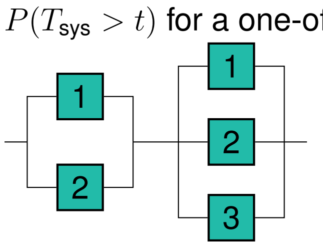
System reliability

We want to find the *system reliability* $P(T_{\text{sys}} > t)$ for a one-of-a-kind system:



$n_1 = 2$
$n_2 = 2$
$n_3 = 1$

The system consists of n_k exchangeable components of types 1, …, K.

Component Lifetimes

The lifetime for each k is assumed as Weibull with fixed shape β:

$$F_k(t \mid \lambda_k) = 1 - e^{-\frac{t^\beta}{\lambda_k}}$$
$$\mathrm{E}[T \mid \lambda_k] = \sqrt[\beta]{\lambda_k}\,\Gamma(1 + 1/\beta)$$

We have information on λ_k from the component manufacturer, but do not fully trust it and model knowledge on λ_k cautiously with a *set of priors* $\mathcal{M}_k^{(0)}$.

Set of Priors

Each $\mathcal{M}_k^{(0)}$ is taken as a set of conjugate inverse Gamma priors. In terms of canonical parameters $n^{(0)}$, $y^{(0)}$,
$\mathcal{M}_k^{(0)} = \{\mathrm{IG}(n_k^{(0)} + 1, n_k^{(0)} y_k^{(0)}) \mid [\underline{n}_k^{(0)}, \overline{n}_k^{(0)}] \times [\underline{y}_k^{(0)}, \overline{y}_k^{(0)}]\}$,
where $y_k^{(0)} = \mathrm{E}[\lambda_k \mid n_k^{(0)}, y_k^{(0)}]$ and $n_k^{(0)} =$ pseudocounts.
The prior parameter set $\Pi_k^{(0)} = [\underline{n}_k^{(0)}, \overline{n}_k^{(0)}] \times [\underline{y}_k^{(0)}, \overline{y}_k^{(0)}]$ allows for more imprecision in case of *prior-data conflict* [2].

Data

We observe the system from startup until t_{now}. For each k, the data $\mathbf{t}^k_{e_k;n_k}$ consists of e_k failure times and $n_k - e_k$ censored observations.
$n_k^{(0)}$ and $y_k^{(0)}$ are updated to $n_k^{(n)}$ and $y_k^{(n)}$ via Bayes' Rule.

Need to minimize over $n_k^{(0)}$'s only, as min must be reached for $\underline{y}_k^{(0)}$'s (lower expected lifetimes = lower component survival probabilities = lower system survival probability).

$$\underline{P}\left(T_{\text{sys}} > t \mid \{n_k^{(0)}, y_k^{(0)}, \mathbf{t}^k_{e_k;n_k}\}^{1:K}\right) = \min_{n_1^{(0)},\ldots,n_K^{(0)}} \sum_{l_1=0}^{n_1-e_1} \cdots \sum_{l_K=0}^{n_K-e_K} \Phi(l_1,\ldots,l_K) \prod_{k=1}^{K} P(C_t^k = l_k \mid n_k^{(0)}, \underline{y}_k^{(0)}, \mathbf{t}^k_{e_k;n_k})$$

Survival signature $\Phi(l_1,\ldots,l_K)$ [1]
$= P(\text{system functions} \mid \{l_k \text{ k's function}\}^{1:K})$

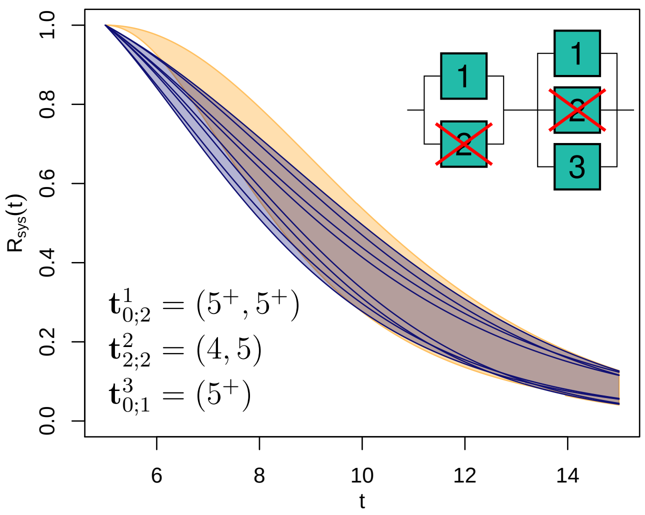
l_1	l_2	l_3	Φ
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0.5
0	2	0	1
0	2	1	1
1	0	0	0
1	0	1	0.5
1	1	1	0.75
⋮	⋮	⋮	⋮

Posterior predictive probability that l_k of the $n_k - e_k$ surviving k's function at time t:

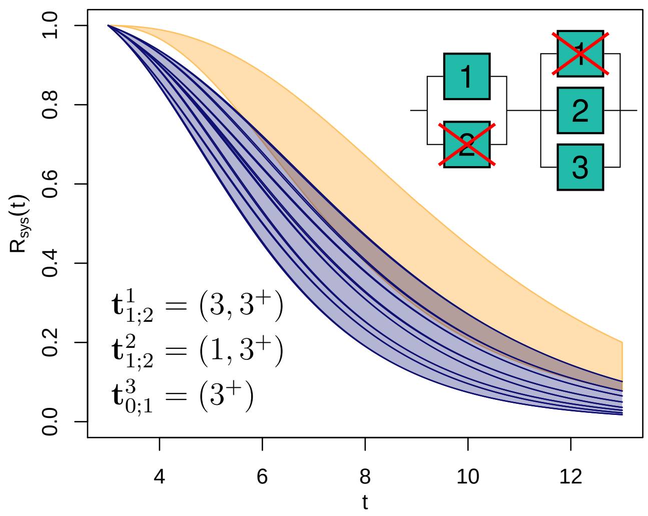
$$\binom{n_k - e_k}{l_k} \int \Big[P_k(T > t \mid T > t_{\text{now}}, \lambda_k)\Big]^{l_k} \times \Big[1 - P_k(T > t \mid T > t_{\text{now}}, \lambda_k)\Big]^{n_k - e_k - l_k} f_{\lambda_k \mid \ldots}(\lambda_k \mid n_k^{(0)}, y_k^{(0)}, \mathbf{t}^k_{e_k;n_k})\, \mathrm{d}\lambda_k$$
$$= \binom{n_k - e_k}{l_k} \sum_{j=0}^{n_k - e_k - l_k} (-1)^j \binom{n_k - e_k - l_k}{j} \left(\frac{n_k^{(n)} y_k^{(n)}}{n_k^{(n)} y_k^{(n)} + (l_k + j)(t^\beta - (t_{\text{now}})^\beta)}\right)^{n_k^{(n)}+1}$$

We assume $\beta = 2$, $\mathrm{E}[T \mid y_1^{(0)}] \in [9, 11]$, $n_1^{(0)} \in [2, 10]$, $\mathrm{E}[T \mid y_2^{(0)}] \in [4, 5]$, $n_2^{(0)} \in [8, 16]$, and $\mathrm{E}[T \mid y_3^{(0)}] \in [9, 11]$, $n_3^{(0)} \in [1, 5]$.

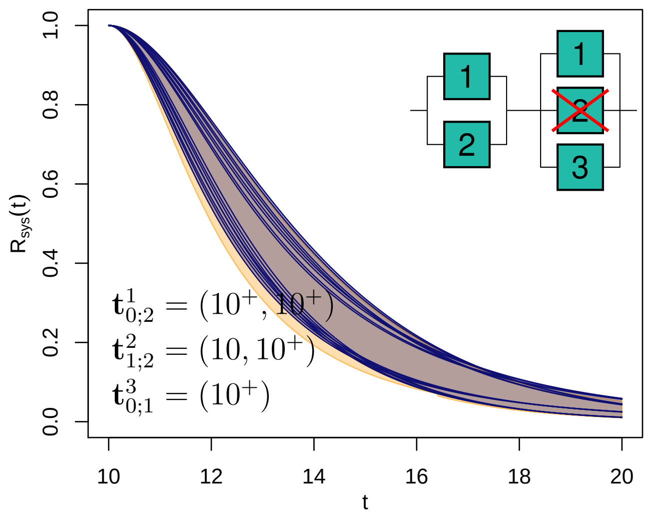
Failure times as expected



Surprisingly early failures



Surprisingly late failures



References

[1] Frank P. A. Coolen and Tahani Coolen-Maturi. Generalizing the signature to systems with multiple types of components. In W. Zamojski, J. Mazurkiewicz, J. Sugier, T. Walkowiak, and J. Kacprzyk, editors, *Complex Systems and Dependability*, volume 170 of *Advances in Intelligent and Soft Computing*, pages 115–130. Springer, 2012.

[2] G. Walter. *Generalized Bayesian Inference under Prior-Data Conflict*. PhD thesis, Department of Statistics, LMU Munich, 2013. http://edoc.ub.uni-muenchen.de/17059.